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Introducing the γ-order Cauchy distribution

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ABSTRACT
The target of this paper is to introduce the γ-order Generalized Cauchy distribution. Based on the γ-order Generalized Normal, $N_\gamma(\mu, \sigma^2 I)$, emerged from Logarithmic Sobolev Inequalities (LSI) a number of γ-order distributions have been already defined. The ratio of two γ-order Generalized Normal defines the γ-order Generalized Cauchy distribution with the probability distribution function evaluated. With $\gamma = 2$ the well-known Cauchy distribution is obtained.

KEYWORDS
γ-order Generalized Normal Distribution, Laplace transformation, Cauchy distribution, Fox-Wright function

1. Introduction

The target of this paper is to offer a γ-order Generalized Cauchy distribution. The γ-order distributions include a number of well-known distributions, see Section 2. The γ-order Generalized Normal was introduced by [8], see also [9]. A compact presentation is here. We say that the random variable X follows the γ-order Generalized Normal distribution, $X \sim N_\gamma(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ when the probability density function (pdf), see [8], is

$$\phi_\gamma(\boldsymbol{x}) = C \exp\left\{-\frac{\gamma-1}{\gamma}[Q(\boldsymbol{x})]^{\frac{\gamma}{2(\gamma-1)}}\right\}, \tag{1}$$

with the normalizing constant to be

$$C = C_p(\boldsymbol{\Sigma}; \gamma) = \frac{1}{\pi^{p/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p\frac{\gamma-1}{\gamma}+1)} \left(\frac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}} \tag{2}$$

and the quadratic form $Q(\boldsymbol{x})$ equal to

$$Q(\boldsymbol{x}) = < \boldsymbol{x} - \boldsymbol{\mu}, \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) >, \ \boldsymbol{\mu} \in \mathbb{R}^p, \ \boldsymbol{\Sigma} \in \mathbb{R}^{p \times p}, \tag{3}$$

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where $< \boldsymbol{x}, \boldsymbol{y} >:= \boldsymbol{x}^T \boldsymbol{y}$ is the square of the Mahalanobis distance of x from distribution $N_\gamma(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\Gamma(\cdot)$ is the Gamma function.

For the probability function (1) the following theorems hold, see [9] and [11], and ensure that the defined family of the γ-order Generalized Normal distribution contains a number of well-known distributions, depending on the value of the shape parameter γ.

Theorem 1.1. *For different values of $\gamma \in \mathbb{R}$-[0,1] and $p \in \mathbb{N}$ the p-variable γ-GN distribution coincides with the Dirac, $D(\boldsymbol{\mu})$, the degenerate vanishing distribution $V(\boldsymbol{\mu})$, the Uniform, $U(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the Normal, $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the Laplace $L(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ as*

$$N_\gamma(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \begin{cases} D(\boldsymbol{\mu}) & \gamma \to 0, p = 1, 2 \\ V(\boldsymbol{\mu}) & \gamma \to 0, p \geq 3 \\ U(\boldsymbol{\mu}, \boldsymbol{\Sigma}) & \gamma \to 1 \\ N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) & \gamma = 2 \\ L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) & \gamma \to \pm\infty \end{cases}$$

Consider the $N_\gamma(\boldsymbol{\mu}, \boldsymbol{I}\sigma^2), \boldsymbol{I} \in \mathbb{R}^{p \times p}$, see [9], with position parameter (mean) $\boldsymbol{\mu}$, positive scale parameter (standard deviation) σ, extra shape parameter γ ($\gamma \in \mathbb{R} - [0, 1]$) and the density function $\phi_\gamma(x; \mu, \sigma^2)$ coming from (1)-(3) and given by

$$\phi_\gamma(x; \mu, \sigma^2) = \frac{\lambda_\gamma}{\sqrt{\pi\sigma^2}} \exp\left\{ -\frac{\gamma - 1}{\gamma} \left(\frac{|x - \mu|}{\sqrt{\sigma^2}} \right)^{\frac{\gamma}{\gamma-1}} \right\}, \tag{4}$$

where $p = 1$ and

$$\lambda_\gamma = \frac{\Gamma(\frac{1}{2} + 1)}{\Gamma(\frac{\gamma-1}{\gamma} + 1)} \left(\frac{\gamma - 1}{\gamma} \right)^{\frac{\gamma-1}{\gamma}}, \quad \gamma_0 = \frac{\gamma - 1}{\gamma}, \quad \gamma_1 = \frac{\gamma}{\gamma - 1}. \tag{5}$$

With [10, Theorem 3.1] Theorem 1.1 is valid for $\phi_\gamma(x; \mu, \sigma^2)$ pdf, i.e. for $\gamma \to 0$, $\phi_\gamma(x)$ coincides with the Dirac distribution and for $\gamma \to 1, \pm\infty$ coincides with the Uniform and Laplace distribution respectively. For $\gamma = 2$ the well-known Normal distribution is achieved.

The Laplace transform of $N_\gamma(\mu, \sigma^2)$ has been stated, evaluated, proved in [7].

Theorem 1.2. *The Laplace transform of $\phi_\gamma(x; \mu, \sigma^2)$ is*

$$\mathcal{L}\phi_\gamma(\xi) = \frac{e^{\xi\mu}}{\Gamma(\gamma_0)} \sum_{j=0}^{\infty} \frac{1}{(2j)!} \left(\xi\sigma(\gamma_1)^{\gamma_0} \right)^{2j} \Gamma\left((2j + 1)\gamma_0 \right), \ \gamma_0 = \frac{\gamma - 1}{\gamma}, \ \gamma_1 = \frac{1}{\gamma_0}. \tag{6}$$

Now, consider the Fox-Wright functions ${}_r\Psi_q(\cdot)$, studied in [1], with applications among others, in fractional calculus, as

$${}_r\Psi_q(z; A, B) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\prod_{k=1}^{r} \Gamma(a_k + j\alpha_k)}{\prod_{l=1}^{q} \Gamma(b_l + j\beta_l)}, \tag{7}$$

where

$$A = [(a_1, \alpha_1), (a_2, \alpha_2), \ldots, (a_r, \alpha_r)], \quad B = [(b_1, \beta_1), (b_2, \beta_2), \ldots, (b_q, \beta_q)].$$

The series in (7) converges for all finite z (has infinite radius of convergence), [1], if

$$\kappa = \sum_{l=1}^{q} \beta_l - \sum_{k=1}^{r} \alpha_k + 1 > 0 \tag{8}$$

and diverges for all $z \neq 0$ when $\kappa < 0$.

Choosing $r = 2, q = 1, a_1 = \alpha_1 = b_1 = 1, a_2 = \gamma_0, \alpha_2 = 2\gamma_0$ and $\beta_1 = 2$ in (7) yields

$$\begin{aligned} {}_2\Psi_1(z; A, B) &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Gamma(a_1 + j\alpha_1)\Gamma(a_2 + j\alpha_2)}{\Gamma(b_1 + j\beta_1)} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Gamma(1 + j)\Gamma(\gamma_0 + 2j\gamma_0)}{\Gamma(1 + 2j)} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{(2j)!} \Gamma((2j + 1)\gamma_0). \end{aligned} \tag{9}$$

Note that (8) is reduced to $\kappa = \beta_1 - \alpha_1 - \alpha_2 + 1 = 2(1 - \gamma_0)$. Therefore $\kappa > 0$ and thus by (8) the series which converges for all finite z whenever $\gamma_0 < 1$ which corresponds to $\gamma > 1$ and accordingly diverges for all $\gamma < 0$.

Therefore due to the above discussion the corollary of Theorem 1.2 is stated with $A_0 = [(1, 1), (\gamma_0, 2\gamma_0)]$, $B_0 = [(1, 2)]$.

Corollary 1.2.0.1. *The Laplace transform of $\phi_\gamma(x; \mu, \sigma^2)$, with $\gamma > 1$, is written as*

$$\mathcal{L}\phi_\gamma(\xi) = \frac{e^{\xi\mu}}{\Gamma(\gamma_0)} {}_2\Psi_1(\xi^2\sigma^2(\gamma_1)^{2\gamma_0}; A_0, B_0). \tag{10}$$

Representation (10) also holds when $\gamma \downarrow 1$.

We also state the following theorem where the evaluation of the cumulative distribution function (cdf) of the standard γ-order Normal distribution is provided, see [11].

Theorem 1.3. *Let $\Phi_\gamma(\cdot)$ denote the cdf of the standard γ-order Normal distribution. Then*

$$\Phi_\gamma(z) = \frac{1}{2} + \frac{\sqrt{\pi}\,\mathrm{sgn}(z)}{2\Gamma(\frac{\gamma-1}{\gamma})\Gamma(\frac{\gamma}{\gamma-1})} \mathrm{Erf}_{\frac{\gamma}{\gamma-1}} \left\{ \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} |z| \right\}. \tag{11}$$

Moreover, the $N_\gamma(\mu, \sigma^2)$ can be adapted to generalize other known distributions such as the truncated Normal, the Chi-square, the Rayleigh and the Maxwell-Boltzmann distributions, see [6], [5].

2. The γ-order distributions

The γ-order Normal distribution, as in (1), has been the basis for the development of more generalizations with main ingredient the Normal $N(\mu, \sigma^2)$ distribution.

Consider $X \sim N_\gamma(\mu, \sigma^2 I)$. The truncated γ-order Normal to the right at $x = \tau$, see [11], is defined as

$$f_{\gamma,\tau^+}(x) = \begin{cases} 0 & \text{if} \quad x > \tau \\ \frac{C}{\Phi_\gamma(\frac{\tau-\mu}{\sigma})\sigma} \exp\{-\frac{\gamma-1}{\gamma}|\frac{x-\mu}{\sigma}|^{\frac{\gamma}{\gamma-1}}\} & \text{if} \quad x \leq \tau \end{cases}$$

and the truncated γ-order Normal to the left at $x = \tau_0$

$$f_{\gamma,\tau_0^-}(x) = \begin{cases} 0 & \text{if} \quad x < \tau_0 \\ \frac{C}{[1-\Phi_\gamma(\frac{\tau_0-\mu}{\sigma})]\sigma} \exp\{-\frac{\gamma-1}{\gamma}|\frac{x-\mu}{\sigma}|^{\frac{\gamma}{\gamma-1}}\} & \text{if} \quad x \geq \tau_0 \end{cases}$$

with C as in (2) and $p = 1$.

Consider the logarithm of the rv X that follows the γ-order Normal, that is, $\ln X \sim N_\gamma(\mu, \sigma^2)$. Then X is said to follow the the γ-order Lognormal distribution, denoted by $LN_\gamma(\mu, \sigma)$, with pdf, [16]

$$f_{LN_\gamma}(x; \mu, \sigma) = \frac{1}{2x\sigma\Gamma(\frac{\gamma-1}{\gamma})\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma}}} \exp\left\{-\frac{\gamma-1}{\gamma}\left(\frac{|\ln x - \mu|}{\sigma}\right)^{\frac{\gamma}{\gamma-1}}\right\}. \tag{12}$$

Consider the sum of the independent identically distributed random variables $Z_i = \frac{X_i-\mu}{\sigma}$, with $Z_i \sim N_\gamma(0,1), i = 1, 2, \ldots, n$ and the sum

$$Y_n = \sum_{i=1}^{n} Z_i^2. \tag{13}$$

Working with the convolution, the γ-order generalized Chi-square distribution was defined in [6], denoted by ${}_\gamma\mathcal{X}_n^2$, where for $\gamma = 2$ the Chi-square can be achieved. The pdf of the produced ${}_\gamma\mathcal{X}_n^2$ is equal to

$$f_{{}_\gamma\mathcal{X}_n^2}(y) = \frac{\gamma_0^{n\gamma_0}}{2\gamma_0\Gamma(n\gamma_0)} y^{\frac{n}{2}-1} \exp\{-\gamma_0(\sqrt{y})^{\gamma_1}\} \tag{14}$$

for any $y > 0$ (when $n = 1$) and any $y \geq 0$ (when $n > 1, n \in \mathbb{N}$). The following theorem is stated and proved in [6] for the Laplace transform of ${}_\gamma\mathcal{X}_n^2$ with

$$\mathcal{L}f_{{}_\gamma\mathcal{X}_n^2}(\xi) = \frac{1}{\Gamma(n\gamma_0)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\xi(\gamma_1)^{2\gamma_0}\right)^j \Gamma((2j+n)\gamma_0). \tag{15}$$

The "two-sided" generalization is the extension of the defined γ-order Chi-square: the γ-order Chi-γ. Indeed, for the independent variables $Z_i, i = 1, 2, \ldots, n$ from

$N_\gamma(0,1)$ i.e. $Z_i \sim N_\gamma(0,1), i = 1,2,\ldots,n$ let us consider the sum

$$Y_n^* = \sum_{i=1}^{n} Z_i^{\gamma_1}, \quad \gamma_1 = \frac{\gamma}{\gamma - 1}. \tag{16}$$

Then the γ-order generalized Chi-γ, ${}_\gamma\mathcal{X}_n^\gamma$ is well-defined, see [5], while for $\gamma = 2$ the Chi-square can be achieved. The pdf of ${}_\gamma\mathcal{X}_n^\gamma$ is proved to be

$$f_{{}_\gamma\mathcal{X}_n^\gamma}(y) = \left(\sqrt{\pi}\frac{\gamma_0^{\gamma_0}}{2\Gamma(\gamma_0+1)}\right)^n \frac{2\gamma_0}{\Gamma(n/2)} y^{n\gamma_0 - 1}\exp\{-\gamma_0 y\}. \tag{17}$$

The Laplace transform of the defined generalized γ-order Chi-γ distribution, ${}_\gamma\mathcal{X}_n^\gamma$, has been evaluated in [5] and is equal to

$$\mathcal{L}\{f_{{}_\gamma\mathcal{X}_n^\gamma}(\xi)\} = \frac{\pi^{n/2}}{2^{n-1}} \frac{\Gamma(1+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)} \frac{\gamma_0^{n\gamma_0}}{n} \left(\frac{1}{\gamma_0 - \xi}\right)^{n\gamma_0}, \quad \xi < \gamma_0. \tag{18}$$

That is, in mathematical terms (18) is a generalized form of the γ-order Chi-square as in (15).

Recall the Rayleigh distribution, see [15] with the applicable framework in Physics. The γ-order Rayleigh distribution, say $\mathcal{R}_\gamma$, can be obtained as, [6],

$$\mathcal{R}_\gamma = \sqrt{\sum_{i=1}^{2} |Z_i|^2} \quad \text{where} \quad Z_i \sim N_\gamma(0,1). \tag{19}$$

If we let $Y \sim \mathcal{R}_\gamma$ then the pdf of Y is equal to, see [6, Theorem 4.1],

$$f_{\mathcal{R}_\gamma}(y) = \frac{\gamma_0^{2\gamma_0 - 1}}{\Gamma(2\gamma_0)} y \exp\{-\gamma_0 y^{\gamma_1}\}, \quad y > 0. \tag{20}$$

Working with the other distribution with application in Physics, recall the Maxwell-Boltzmann distribution, see [12]. The γ-order Maxwell-Boltzmann distribution, say $\mathcal{MB}_\gamma$, can be obtained as

$$\mathcal{MB}_\gamma = \sqrt{\sum_{i=1}^{3} |Z_i|^2} \quad \text{where} \quad Z_i \sim N_\gamma(0,1). \tag{21}$$

If we let $Y \sim \mathcal{MB}_\gamma$ then the pdf of Y is equal to, see [6, Theorem 4.2],

$$f_{\mathcal{MB}_\gamma}(y) = \frac{\gamma_0^{3\gamma_0 - 1}}{\Gamma(3\gamma_0)} y^2 \exp\{-\gamma_0 y^{\gamma_1}\}, \quad y > 0. \tag{22}$$

We have already a set of five different γ-order families of distributions as presented above. In this paper the γ-order Cauchy distribution is defined and studied.

3. The γ-order Cauchy distribution

Let the rvs X, Y coming from the standard Normal, $X \sim N(0,1), Y \sim N(0,1)$. Then their ratio $C = \frac{X}{Y}$ follows the Cauchy $Cau(0,1)$ distribution. The cdf of C is

$$F_{\frac{X}{Y}}(c) = \mathbb{P}(\frac{X}{Y} \leq c) = \mathbb{P}(\frac{X}{|Y|} \leq c) = \mathbb{P}(X \leq c|Y|),$$

where the symmetric property of the Normal distribution was used. Therefore,

$$\begin{aligned} F_{\frac{X}{Y}}(c) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \left(\int_{-\infty}^{c|y|} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \Phi(c|y|) dy \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y^2/2} \Phi(cy) dy, \end{aligned} \tag{23}$$

where $\Phi(\cdot)$ denotes the cdf of the standard Normal. In the last step before (23) the fact that the integrated function is even was applied. From the cdf of the Cauchy random variable, (23), the corresponding pdf can be evaluated. Therefore

$$\begin{aligned} f_{\frac{X}{Y}}(c) = \frac{d}{dc} F_{\frac{X}{Y}}(c) &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y^2/2} \frac{d}{dc} \Phi(cy) dy \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} y e^{-y^2/2} \phi(cy) dy, \end{aligned}$$

where the interchange of integration and differentiation was applied (Leibniz integral rule) and $\phi(\cdot)$ denotes the pdf of the standard Normal. Therefore

$$\begin{aligned} f_{\frac{X}{Y}}(c) &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} y e^{-y^2/2} \frac{1}{\sqrt{2\pi}} e^{-(cy)^2/2} dy \\ &= \frac{1}{\pi} \int_{0}^{\infty} y \exp\{-\frac{1}{2} y^2 (1+c^2)\} dy \\ &= \frac{1}{\pi(1+c^2)} \int_{0}^{\infty} \exp\{-w\} dw \\ &= \frac{1}{\pi(1+c^2)} := f_{Cau}(c), \end{aligned} \tag{24}$$

where the transformation $w = \frac{1}{2} y^2 (1+c^2)$ was applied. Following the above line of thought, the γ-order Generalized Cauchy distribution, $Cau_\gamma(\cdot)$, can be defined.

Let $X \sim N_\gamma(0,1), Y \sim N_\gamma(0,1)$. Then for their ratio $Cau_\gamma = \frac{X}{Y}$ it holds that

$$F_{\frac{X}{Y}}(c) = \mathbb{P}(X \leq c|Y|).$$

Therefore,

$$\begin{aligned} F_{\frac{X}{Y}}(c) &= \frac{\lambda_\gamma}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\gamma_0 |y|^{\gamma_1}} \left(\int_{-\infty}^{c|y|} \frac{\lambda_\gamma}{\sqrt{\pi}} e^{-\gamma_0 |x|^{\gamma_1}} dx \right) dy \\ &= \frac{\lambda_\gamma}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\gamma_0 |y|^{\gamma_1}} \Phi_\gamma(c|y|) dy \\ &= \frac{2\lambda_\gamma}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\gamma_0 y^{\gamma_1}} \Phi_\gamma(cy) dy := F_{Cau_\gamma}(c), \end{aligned} \tag{25}$$

where in the last step the fact that the integrated function is even was applied. Therefore the pdf of $Cau_\gamma(1)$ can be evaluated, as is stated and proved in the following theorem.

Theorem 3.1. *Let $X, Y \sim N_\gamma(0,1)$. Then their ratio $Cau_\gamma(1) = \frac{X}{Y}$ follows the γ-order Generalized Cauchy distribution with pdf*

$$f_{Cau_\gamma}(x) = \frac{K_\gamma \sigma}{(1+|x|^{\gamma_1})^{2\gamma_0}}, \ \gamma_0 = \frac{\gamma-1}{\gamma}, \ \gamma_1 = \frac{1}{\gamma_0}, \tag{26}$$

where

$$K_\gamma := \frac{\Gamma(2\gamma_0)}{2\Gamma(\gamma_0)\Gamma(\gamma_0+1)}, \ \gamma_0 = \frac{\gamma-1}{\gamma}. \tag{27}$$

Proof. The cdf of the γ-order Cauchy random variable is obtained in (25) and the derivative of (25) provides the corresponding pdf. It is

$$\begin{aligned} f_{\frac{X}{Y}}(c) = \frac{d}{dc} F_{\frac{X}{Y}}(c) &= \frac{2\lambda_\gamma}{\sqrt{\pi}} \int_0^\infty e^{-\gamma_0 y^{\gamma_1}} \frac{d}{dc} \Phi_\gamma(cy) dy \\ &= \frac{2\lambda_\gamma}{\sqrt{\pi}} \int_0^\infty y e^{-\gamma_0 y^{\gamma_1}} \phi_\gamma(cy) dy, \end{aligned}$$

where $\phi_\gamma(\cdot)$ denotes the pdf of the γ-order standard Normal, by (4) with $\mu = 0$ and $\sigma = 1, \phi_\gamma(x; 0, 1)$. Therefore

$$\begin{aligned} f_{\frac{X}{Y}}(c) &= \frac{2\lambda_\gamma}{\sqrt{\pi}} \int_0^\infty y e^{-\gamma_0 y^{\gamma_1}} \frac{\lambda_\gamma}{\sqrt{\pi}} e^{-\gamma_0 |cy|^{\gamma_1}} dy \\ &= \frac{2(\lambda_\gamma)^2}{\pi} \int_0^\infty y \exp\{-\gamma_0 y^{\gamma_1}(1+|c|^{\gamma_1})\} dy \\ &= \frac{2(\lambda_\gamma)^2}{\pi(1+|c|^{\gamma_1})} \int_0^\infty \left(\frac{w}{\gamma_0(1+|c|^{\gamma_1})} \right)^{2\gamma_0 - 1} \exp\{-w\} dw, \end{aligned} \tag{28}$$

where the transformation $w = \gamma_0 y^{\gamma_1}(1+|c|^{\gamma_1})$ was applied and the corresponding differential evaluated and it is an extension to the w-transformation in (24). Recalling the definition of λ_γ as in (5) and working the necessary calculation in (28) applying the definition of the $\Gamma(\cdot)$ function for the integral below results to

$$\begin{aligned} f_{\frac{X}{Y}}(c) &= \frac{2(\frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\gamma_0+1)}\gamma_0^{\gamma_0})^2}{\pi\gamma_0^{2\gamma_0-1}(1+|c|^{\gamma_1})^{2\gamma_0}} \int_0^\infty w^{2\gamma_0-1}\exp\{-w\}dw \\ &= \frac{\gamma_0}{2\Gamma(\gamma_0+1)\Gamma(\gamma_0+1)(1+|c|^{\gamma_1})^{2\gamma_0}}\Gamma(2\gamma_0) \\ &= \frac{\Gamma(2\gamma_0)}{2\Gamma(\gamma_0)\Gamma(\gamma_0+1)(1+|c|^{\gamma_1})^{2\gamma_0}}. \end{aligned} \tag{29}$$

Thus we form the pdf of $Cau_\gamma(1)$

$$f_{Cau_\gamma}(c) = \frac{\Gamma(2\gamma_0)}{2\Gamma(\gamma_0)\Gamma(\gamma_0+1)(1+|c|^{\gamma_1})^{2\gamma_0}} = \frac{K_\gamma}{(1+|c|^{\gamma_1})^{2\gamma_0}}, \tag{30}$$

where (27) was applied which is exactly (26) when the argument c is replaced by x. □

The values of K_γ are presented in Table **??** and provide that $0.25 < K_\gamma < 0.50$ for $\gamma > 1$ while when approaching 0 from the left they are not bounded (NaN). Notice that for $\gamma < -1$ it is $0.5 < K_\gamma < 1.34$ and K_γ is, in principle, an increasing function of γ. In Figure 1 the pdf of the standard γ-order Cauchy distribution is illustrated for different values of γ, recall Theorem 1.1.

Table 1. Values of K_γ for different γ.

γ		1.1	1.8	2	2.2	3	50
K_γ		0.2530	0.3053	0.3183	0.3299	0.3652	0.4901
γ	-0.01	-1.1	-1.8	-2	-2.2	-3	-50
K_γ	NaN	1.3496	0.9028	0.8488	0.8073	0.7076	0.5101

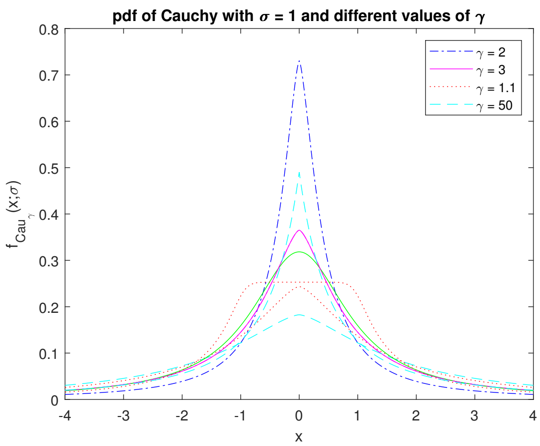

Figure 1. Plots of the pdf of the standard γ-order Cauchy distribution.

Corollary 3.1.0.1. *Let $X, Y \sim N_2(0,1)$. Then their ratio $Cau_2(1) = \frac{X}{Y}$ follows the*

2-*order Generalized Cauchy distribution with pdf*

$$f_{Cau_2}(x) = \frac{1}{\pi(1+x^2)}, \tag{31}$$

which is the known Cauchy distribution, see (24).

Proof. For $\gamma = 2$ in definition (5) $\gamma_0 = 1/2, \gamma_1 = 2$ and by (26) one arrives to (31). □

Based on the above discussion, the ratio of two γ-order Normal distributions with scale parameter σ is presented in Theorem 3.2 below. Before the statement and proof of the theorem an auxiliary lemma follows.

Lemma 3.1.1. *The following representation is true*

$$\pi \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{(2j)!} \Gamma((2j+1)\gamma_0)\Gamma(-(2j-1)\gamma_0) = {}_5\Psi_1(z; A_1, B_1) \tag{32}$$

where

$$A_1 = [(\gamma_0, 2\gamma_0), (\gamma_0, -2\gamma_0), (1,1), (1/2, 1), (1/2, -1)], \quad B_1 = [(1,2)].$$

The series (32) converges for all values of z.

Proof. Consider the Fox-Wright functions ${}_r\Psi_q(\cdot)$, as in (7), and further studied in [2]. Taking $r = 5, q = 1, a_3 = \alpha_3 = \alpha_4 = b_1 = 1, a_1 = a_2 = \gamma_0, \alpha_1 = 2\gamma_0, \alpha_2 = -2\gamma_0, \alpha_5 = -1, a_4 = a_5 = 1/2$ and $\beta_1 = 2$ in (7) yields

$$\begin{aligned} {}_5\Psi_1(z; A_1, B_1) &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Gamma(a_1 + j\alpha_1)\Gamma(a_2 + j\alpha_2)\Gamma(a_3 + j\alpha_3)\Gamma(a_4 + j\alpha_4)\Gamma(a_5 + j\alpha_5)}{\Gamma(b_1 + j\beta_1)} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Gamma(\gamma_0 + j2\gamma_0)\Gamma(\gamma_0 - j2\gamma_0)\Gamma(1+j)\Gamma(1/2+j)\Gamma(1/2-j)}{\Gamma(1+2j)} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{(2j)!} \Gamma((2j+1)\gamma_0)\Gamma((1-2j)\gamma_0)\Gamma(1/2+j)\Gamma(1/2-j). \end{aligned} \tag{33}$$

Applying Euler's reflection formula

$$\Gamma(x+1)\Gamma(-x) = \frac{\pi}{\sin(\pi(x+1))} \tag{34}$$

for $x = j - 1/2$ implies that

$${}_5\Psi_1(z; A_1, B_1) = \pi \sum_{j=0}^{\infty} (-1)^j \frac{z^j}{(2j)!} \Gamma((2j+1)\gamma_0)\Gamma((1-2j)\gamma_0).$$

Note that

$$\kappa = \beta_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 + 1 = 2 - 2\gamma_0 + 2\gamma_0 - 1 - 1 + 1 + 1 = 2 > 0.$$

Therefore by [2, Theorem 2] the series converges for all z, that is for all values of γ. $\square$

Theorem 3.2. *Let the rvs $X, Y \sim N_\gamma(0, \sigma^2)$. Then their ratio $Cau_\gamma(\sigma) = \frac{X}{Y}$ follows the γ-order Generalized Cauchy distribution with scale parameter σ and pdf*

$$f_{Cau_\gamma}(x;\sigma) = \frac{K_\gamma \sigma}{(\sigma^{\gamma_1} + |x|^{\gamma_1})^{2\gamma_0}} \tag{35}$$

with K_γ as in (27) and Fourier transform of (35)

$$\mathcal{F}[f_{Cau_\gamma}(x;\sigma)](\omega) = \hat{f}_{Cau_\gamma}(\omega) = \frac{1}{\pi\Gamma^2(\gamma_0)}\,{}_5\Psi_1(\sigma^2\omega^2; A_2, B_2), \tag{36}$$

with $A_2 = [(\gamma_0, 2\gamma_0), (\gamma_0, -2\gamma_0), (1,1), (1/2, 1), (1/2, -1)]$ and $B_2 = [(1, 2)]$.

Proof. Let $X \sim N_\gamma(0, \sigma^2), Y \sim N_\gamma(0, \sigma^2)$. Then the rv of their ratio $Cau_\gamma(\sigma) = \frac{X}{Y}$ has a cdf, following the line of though that resulted in (25), given by

$$F_{Cau_\gamma}(x;\sigma) = \frac{2\lambda_\gamma}{\sigma\sqrt{\pi}} \int_0^\infty e^{-\gamma_0 (y/\sigma)^{\gamma_1}} \Phi_\gamma(xy/\sigma)dy \tag{37}$$

and the corresponding pdf reads

$$\begin{aligned}
f_{Cau_\gamma}(x;\sigma) &= \frac{2\lambda_\gamma}{\sigma\sqrt{\pi}} \int_0^\infty \frac{y}{\sigma} e^{-\gamma_0 (y/\sigma)^{\gamma_1}} \frac{\lambda_\gamma}{\sigma\sqrt{\pi}} e^{-\gamma_0 |xy/\sigma^2|^{\gamma_1}} dy \\
&= \frac{2(\lambda_\gamma)^2}{\sigma^2\pi} \int_0^\infty \frac{y}{\sigma} \exp\{-\gamma_0 (y/\sigma)^{\gamma_1}(1 + |x/\sigma|^{\gamma_1})\} dy \\
&= \frac{2(\lambda_\gamma)^2}{\sigma\pi(1 + |x/\sigma|^{\gamma_1})} \int_0^\infty \left(\frac{w}{\gamma_0(1 + |x/\sigma|^{\gamma_1})}\right)^{2\gamma_0 - 1} \exp\{-w\}dw \\
&= \frac{\Gamma(2\gamma_0)}{2\sigma\Gamma(\gamma_0)\Gamma(\gamma_0 + 1)(1 + |x/\sigma|^{\gamma_1})^{2\gamma_0}},
\end{aligned} \tag{38}$$

where the transformation $w = \gamma_0 y^{\gamma_1}(1 + |x/\sigma|^{\gamma_1})/\sigma^{\gamma_1}$ was applied, recall (28) and (24) for the extension steps we followed. Relation (38) is the form of the γ-order Cauchy distribution involving the scale parameter σ.

Let the rv X follow a γ-order Cauchy distribution with parameter σ, that is $X \sim Cau_\gamma(\sigma)$. Then by (38) the pdf of X is written as (35), where K_γ as in (27).

Now the evaluation of the Fourier transform of the γ-order Cauchy pdf follows. It is

$$
\begin{aligned}
\mathcal{F}[f_{Cau_\gamma}(x;\sigma)](\omega) &= \int_{-\infty}^{\infty} f_{Cau_\gamma}(x;\sigma)e^{-ix\omega}dx \\
&= \frac{\Gamma(2\gamma_0)}{2\sigma\Gamma(\gamma_0)\Gamma(\gamma_0+1)} \int_{-\infty}^{\infty} \frac{1}{(1+|x/\sigma|^{\gamma_1})^{2\gamma_0}} e^{-ix\omega}dx \\
&= \frac{K_\gamma}{\sigma}\left[\int_{-\infty}^{0} \frac{1}{(1+(-x/\sigma)^{\gamma_1})^{2\gamma_0}} e^{-ix\omega}dx + \int_{0}^{\infty} \frac{1}{(1+(x/\sigma)^{\gamma_1})^{2\gamma_0}} e^{-ix\omega}dx\right] \\
&= \frac{K_\gamma}{\sigma}\int_{0}^{\infty} \frac{1}{(1+(x/\sigma)^{\gamma_1})^{2\gamma_0}} (e^{ix\omega}+e^{-ix\omega})dx \\
&= \frac{2K_\gamma}{\sigma}\int_{0}^{\infty} \frac{\cos(x\omega)}{(1+(x/\sigma)^{\gamma_1})^{2\gamma_0}} dx
\end{aligned}
$$

or

$$
\mathcal{F}[f_{Cau_\gamma}(x;\sigma)](\omega) = 2K_\gamma \int_{0}^{\infty} \frac{\cos(x\sigma\omega)}{(1+x^{\gamma_1})^{2\gamma_0}} dx \tag{39}
$$

Now the integral in (39), say I_n, is expressed as

$$
\begin{aligned}
I_n &= \int_{0}^{\infty} \frac{1}{(1+x^{\gamma_1})^{2\gamma_0}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n (x\sigma\omega)^{2n}}{(2n)!}\right) dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\sigma\omega)^{2n}}{(2n)!} \int_{0}^{\infty} \frac{x^{2n}}{(1+x^{\gamma_1})^{2\gamma_0}} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\sigma\omega)^{2n}}{(2n)!} \int_{0}^{\infty} \frac{(y^{2\gamma_0})^{2n}}{(1+y^2)^{2\gamma_0}} 2\gamma_0 y^{2\gamma_0-1} dy
\end{aligned}
$$

where the transformation with $x^{\gamma_1} = y^2$ was applied. Now, the last integral above, say J_n, is evaluated as

$$
\begin{aligned}
J_n &= 2\gamma_0 \int_{0}^{\infty} \frac{(y^{2\gamma_0})^{2n+1}}{(1+y^2)^{2\gamma_0}} \frac{1}{y} dy \\
&= 2\gamma_0 \int_{0}^{\pi/2} (\cos\omega)^{-4n\gamma_0+2\gamma_0-1} (\sin\omega)^{4n\gamma_0+2\gamma_0-1} d\omega \\
&= \gamma_0 2 \int_{0}^{\pi/2} (\sin\omega)^{2(2n\gamma_0+\gamma_0)-1} (\cos\omega)^{2(-2n\gamma_0+\gamma_0)-1} d\omega \\
&= \gamma_0 B(2n\gamma_0+\gamma_0, -2n\gamma_0+\gamma_0)
\end{aligned} \tag{40}
$$

with $y = \tan\omega$. Therefore by (39) it is

$$\begin{aligned}
\mathcal{F}[f_{Cau_\gamma}(x;\sigma)](\omega) &= 2\gamma_0 K_\gamma \sum_{n=0}^{\infty} \frac{(-1)^n(\sigma\omega)^{2n}}{(2n)!} B(2n\gamma_0+\gamma_0, -2n\gamma_0+\gamma_0) \\
&= \frac{\gamma_0\Gamma(2\gamma_0)}{\Gamma(\gamma_0)\Gamma(\gamma_0+1)} \sum_{n=0}^{\infty} \frac{(-1)^n(\sigma\omega)^{2n}}{(2n)!} B((2n+1)\gamma_0, (-2n+1)\gamma_0) \\
&= \frac{\gamma_0\Gamma(2\gamma_0)}{\Gamma(\gamma_0)\Gamma(\gamma_0+1)} \sum_{n=0}^{\infty} \frac{(-1)^n(\sigma\omega)^{2n}}{(2n)!} \frac{\Gamma((2n+1)\gamma_0)\Gamma((1-2n)\gamma_0)}{\Gamma(2\gamma_0)} \\
&= \frac{1}{\Gamma^2(\gamma_0)} \sum_{n=0}^{\infty} \frac{(-1)^n(\sigma^2\omega^2)^n}{(2n)!} \Gamma((2n+1)\gamma_0)\Gamma(-(2n-1)\gamma_0). \quad (41)
\end{aligned}$$

Applying Lemma 3.1.1 for the series in (41) implies

$$\mathcal{F}[f_{Cau_\gamma}(x;\sigma)](\omega) = \frac{1}{\pi\Gamma^2(\gamma_0)} {}_5\Psi_1(\sigma^2\omega^2; A_2, B_2).$$

Thus, the Fourier transform of the Generalized γ-order Cauchy is written as (36). □

Corollary 3.2.0.1. *Let $X, Y \sim N_2(0,\sigma^2)$. Then their ratio $Cau_2(\sigma) = \frac{X}{Y}$ follows the 2-order Generalized Cauchy distribution with scale parameter σ with pdf*

$$f_{Cau_2}(x;\sigma) = \frac{\sigma}{\pi(\sigma^2+x^2)} \quad (42)$$

and Fourier transform as in (36)

$$\mathcal{F}[f_{Cau_2}(x;\sigma)](\omega) = \hat{f}_{Cau_2}(\omega) = e^{-\sigma|\omega|}. \quad (43)$$

Proof. For $\gamma = 2$ then $\gamma_0 = 1/2$ and $\gamma_1 = 1/2$ and $K_2 = \frac{1}{\pi}$ thus (35) implies (42). For the Fourier transform use that $\Gamma(1/2) = \sqrt{\pi}$ in (36) with $A = [(1/2,1),(1/2,-1),(1,1),(1/2,1),(1/2,-1)]$ and $B = [(1,2)]$ to get that

$$\begin{aligned}
\hat{f}_{Cau_2}(\omega) = \frac{1}{\pi^2} {}_5\Psi_1(\sigma^2\omega^2; A, B) &= \frac{1}{\pi^2} \sum_{j=0}^{\infty} \frac{(\sigma^2\omega^2)^j}{(2j)!} [\Gamma(j+1/2)\Gamma(1/2-j)]^2 \\
&= \frac{1}{\pi^2} \sum_{j=0}^{\infty} \frac{(\sigma\omega)^{2j}}{(2j)!} \left[\frac{\pi}{\sin(\pi(j+1/2))}\right]^2 \\
&= \frac{1}{\pi^2} \sum_{j=0}^{\infty} \frac{(\sigma\omega)^{2j}}{(2j)!} [(-1)^j\pi]^2 = \cosh(-\sigma\omega).
\end{aligned}$$

Therefore since the argument of the exponential function should be negative we find that

$$\hat{f}_{Cau_2}(\omega) = e^{-\sigma|\omega|}. \quad (44)$$

Alternatively, by (39), it is

$$\begin{aligned}\mathcal{F}[f_{Cau_2}(x;\sigma)](\omega) &= \frac{2}{\pi}\int_0^{\infty}\frac{\cos(x\sigma\omega)}{1+x^2}dx \\ &= \frac{2}{\pi}\frac{\pi}{2}e^{-|\sigma\omega|} = e^{-\sigma|\omega|},\end{aligned}$$

where the evaluation of the integral can be found for instance in [14, Sec. 45.3.20]. □

Figure 2 provides the pdf of the γ-order Cauchy distribution for different values of γ and σ. Note that for γ close to 1 the uniform behavior is apparent, while for large values of γ the behavior of Laplace distribution is evident, recall Theorem 1.1.

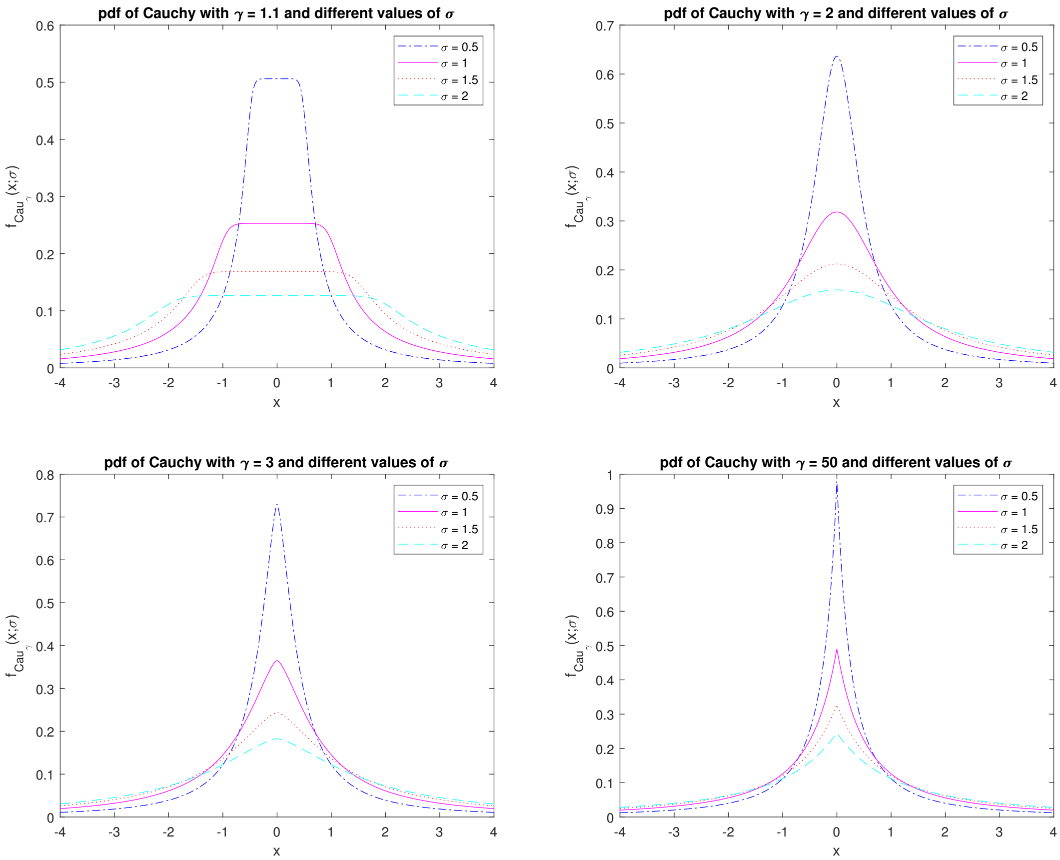

Figure 2. Plots of the pdf of the γ-order Cauchy distribution.

In the following the distribution of the inverse of the rv X is evaluated.

Proposition 1. *Let $X \sim Cau_\gamma(\sigma)$. Then the inverse of the rv X is such that $X^{-1} \sim Cau_\gamma(1/\sigma)$.*

Proof. Let $Y = g(X) = \frac{1}{X}$. Then this transformation implies that $X = g^{-1}(Y) = \frac{1}{Y}$

with $\frac{dX}{dY} = -Y^{-2}$. Therefore

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right| \\
&= \frac{\Gamma(2\gamma_0)\sigma}{2\Gamma(\gamma_0)\Gamma(\gamma_0+1)(\sigma^{\gamma_1} + |\frac{1}{y}|^{\gamma_1})^{2\gamma_0}} y^{-2} \\
&= \frac{K_\gamma \sigma}{\sigma^{2\gamma_1\gamma_0}(|y|^{\gamma_1} + \frac{1}{\sigma^{\gamma_1}})^{2\gamma_0}} \frac{y^{2\gamma_1\gamma_0}}{y^2} = \frac{K_\gamma \frac{1}{\sigma}}{(|y|^{\gamma_1} + (\frac{1}{\sigma})^{\gamma_1})^{2\gamma_0}},
\end{aligned} \tag{45}
$$

where the fact that γ_0 and γ_1 are inverse numbers was applied. By (45) it is understood that $Y \sim Cau_\gamma(1/\sigma)$, that is the inverse of a γ-order Cauchy distribution with parameter σ is a γ-order Cauchy distribution with parameter $1/\sigma$. □

Theorem 3.3. *The Half-γ-order Cauchy distribution with scale parameter σ has pdf*

$$
1/2Cau_\gamma(x;\sigma) = \frac{2K_\gamma \sigma}{(\sigma^{\gamma_1} + x^{\gamma_1})^{2\gamma_0}}, \qquad x \geq 0, \tag{46}
$$

with K_γ as in (27). For $\gamma = 2$ it reduces to the Half-order Cauchy distribution

$$
1/2Cau_2(x;\sigma) = \frac{2}{\pi\sigma} \frac{1}{\left[1 + (\frac{x}{\sigma})^2\right]}, \qquad x \geq 0.
$$

Proof. Just note that the defined distribution is the truncated γ-order Cauchy distribution, therefore it is defined only for values greater or equal to the location of the peak which here is assumed to be zero. □

See Figure 3 for an illustration of the pdf for $1/2Cau_\gamma(x;\sigma)$ with relation to Figure 1, as the Half-γ-order Cauchy distribution is a truncated γ-order Cauchy distribution.

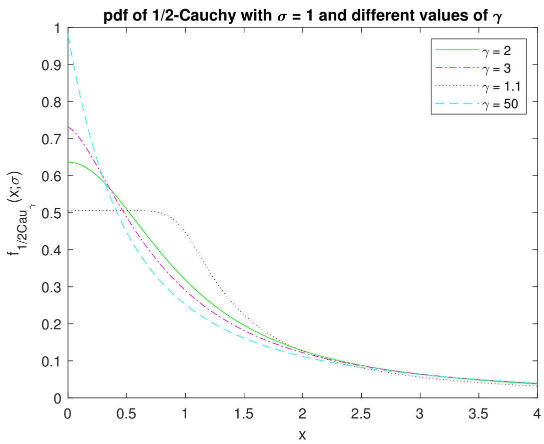

Figure 3. Plots of the pdf of the standard Half-γ-order Cauchy distribution.

Example 3.1. *Consider the transformation $Y = nX + m, n > 0, m \in \mathbb{R}$ where $X \sim Cau_\gamma(\sigma)$. Then, the rv Y follows a γ-order Cauchy distribution with parameter $n\sigma$.*

Proof. The transformation $g(x) = nx+m$ is strictly monotone with inverse $g^{-1}(y) = \frac{y-m}{n}$ and $\frac{dg^{-1}(y)}{dy} = \frac{1}{n}$. Therefore following a similar approach as in Proposition 1 and using (27) it follows that

$$\begin{aligned} f_Y(y) &= f_X\left(\frac{y-m}{n}\right)\left|\frac{dg^{-1}(y)}{dy}\right| \\ &= \frac{K_\gamma \sigma}{(\sigma^{\gamma_1} + |\frac{y-m}{n}|^{\gamma_1})^{2\gamma_0}} n^{-1} \\ &= \frac{K_\gamma \sigma}{n} \frac{n^{2\gamma_1\gamma_0}}{(|y-m|^{\gamma_1} + (n\sigma)^{\gamma_1})^{2\gamma_0}} \\ &= K_\gamma \frac{n\sigma}{(|y-m|^{\gamma_1} + (n\sigma)^{\gamma_1})^{2\gamma_0}}, \end{aligned}$$

which is a γ-order Cauchy distribution with scale parameter $n\sigma$ and location parameter m. □

4. Concluding remarks

In principle "ratios" need a particular treatment in Statistics, either in ratio estimates in sampling or to calibration problems, [3], among other applications as Relative Risk etc. This paper worked on the ratio of two rvs coming from the γ-order Generalized Normal and a γ-order Generalized Cauchy was obtained. With the value of the shape parameter $\gamma = 2$ the typical Cauchy distribution was obtained, as in all the introduced γ-order distributions. The truncated nonzero Cauchy, for chosen values greater or equal to the location of the peak, was also obtained. The mean and variance are undefined to (Half-)Cauchy distribution, and in the case that the sample size $\nu = 1$ we are working with the (Half-)Student distribution. The easiest case of Half-distribution is the distribution of $Y = |X|$ when $X \sim N(\mu, \sigma^2)$, the Half-Normal distribution.

One could think that the Fox-Wright psi function, might be a really new approach, but it has been already used by the definition of the Half-Normal distribution. Therefore, we used the well known ingredients to work and prove the pdf for the γ-order Cauchy and the Half-Cauchy.

In principle the Cauchy distribution is applied in Finance to represent deviation in returns from the assumed correct predictive model. We need heavy-tailed models and the shape parameter can offer such possibilities to escape from the light-tailed distributions. The typical Normal is such a case, but with the shape parameter γ, we can obtain the fat-tailed distributions from the family of the γ-order Generalized Normal. The Cauchy distribution is a heavy-tailed distribution and the γ-order Generalized one can be a "heavier" one.

Notice the influence of the Uniform distribution, when $\gamma \to 1$, the Laplace distribution, $\gamma \to \infty$, practically $\gamma = 50$, see Figure 2, the first and the last.

Due to [13] if $X \sim Cau(\sigma)$ then $\frac{\alpha X+\beta}{\gamma X+\delta} \sim Cau(\frac{\alpha\sigma+\beta}{\gamma\sigma+\delta})$ and this result can be extended to γ-order Generalized Normal, with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We think that the γ-order Generalized Cauchy can be proved useful to applications as the γ-order Generalized Normal has been already used, [4].

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